Consequences of the Cauchy Integral formula

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1) <u>Morera Theorem.</u>

Let f be continuous in a region Ω.

<u>Assume</u> that for any $z_0 \in \Omega$ $\exists B(z_0,r) \subset \Omega$ with the following property:
if $\partial R \subset B(z_0,r)$ - rectangle then $\oint_{\partial R} f(z)\,dz = 0$.

<u>Then</u> $f \in A(\Omega)$.

<u>Proof.</u> Fix $z_0 \in \Omega$. We know that $\exists F \in A(B(z_0,r))$ such that $F'(z) = f(z)$ $\forall z \in B(z_0,r)$. But F is infinitely differentiable $\Rightarrow$ $f'(z_0)$ exists.
It is true for any $z_0 \Rightarrow f \in A(\Omega)$ ∎

<u>Remarks</u>:

a) The condition of Theorem is necessary, by Cauchy-Goursat. Moreover, it holds $\forall B(z_0,r) \subset \Omega$.

b) An easier to state, but weaker form (no longer necessary!):
$f \in C(\Omega)$, $\forall \gamma \subset \Omega$, closed $\oint_\gamma f(z)\,dz = 0 \Rightarrow f \in A(\Omega)$.

2) <u>Theorem</u> (Cauchy inequality). Assume that $f \in A(B(z_0,R))$, $|f(z)| \le M$ $\forall z \in B(z_0,R)$.

<u>Then</u> $|f^{(n)}(z_0)| \le M\, n!\, R^{-n}$

<u>Proof.</u> By Cauchy formula for derivative, $\forall r < R$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}, \quad \text{so} \quad |f^{(n)}(z_0)| \le \frac{n!}{2\pi}\cdot 2\pi r \cdot \frac{M}{r^{n+1}} = M n!\, r^{-n} \ \forall r < R$$ ∎

Corollary $f \in A(\Omega)$, (Ω - a region). $K \subset \Omega$ - compact.
Then $\forall n \; \exists C(K,n): \; \forall z \in K \quad |f^{(n)}(z)| \le C(K,n)$.

Proof For $z \in K$, let $r_z := \operatorname{dist}(z, \partial\Omega)/4$

$K \subset \bigcup_{z \in K} B(z, r_z)$, so $\exists z_1, \dots, z_n: \; K \subset B(z_1, r_{z_1}) \cup \dots B(z_n, r_{z_n})$.

Consider $\tilde{K} := \bigcup_{j=1}^{n} \overline{B(z_j, 2r_{z_j})}$ - compact, as a finite union of closed disks.

So $\exists C: \; |f(w)| \le C \quad \forall w \in \tilde{K}$

Let $r = \min(r_{z_1}, \dots, r_{z_n})$. Then $z \in K \Rightarrow \exists j: |z - z_j| < r_{z_j} \Rightarrow$
$B(z,r) \subset B(z_j, r_{z_j} + r) \subset B(z_j, 2r_{z_j}) \subset \tilde{K}$, $\forall w \in B(z,r)$, $|f(w)| \le C$.

So $|f^{(n)}(z)| \le n!\, C\, r^{-n} =: C(K,n)$ - does not depend on z!

Cauchy inequality

Bonus question (+1): As stated, the corollary does not require Cauchy inequalities.
1) Prove it without them.
2) State and prove stronger version, using Cauchy inequality.

3) 

Joseph Liouville

Theorem (Liouville) Any bounded entire function is identically constant.

Reminder f is called entire if $f \in A(\mathbb{C})$.

Proof. Let f be bounded, $|f(z)| \le M \ \forall z \in \mathbb{C}$.
Then $\forall z_0 \in \mathbb{C}, \forall R>0 \quad f \in \mathcal{A}(B(z_0, R))$.
So $|f'(z_0)| \le \frac{M}{R} \quad \forall R>0$ (by Cauchy).
So $\forall z_0 \in \mathbb{C}, \quad f'(z_0) = 0 \implies f \equiv \text{const}$ ∎

Theorem (Fundamental Theorem of Algebra).
Every polynomial of degree $d>0$ has at least one zero.
Proof Let $p(z) = a_d z^d + \ldots + a_0$. Assume $\forall z \in \mathbb{C}$, $p(z) \ne 0$.
Then $\frac{1}{p(z)} \in \mathcal{A}(\mathbb{C})$. $\lim\limits_{z\to\infty} \frac{1}{|p(z)|} = \lim\limits_{z\to\infty} \left|\frac{1}{z^d}\right| \cdot \lim\limits_{z\to\infty} \frac{1}{\left|a_d + \frac{a_{d-1}}{z} + \ldots\right|} = 0 \cdot \frac{1}{|a_d|} = 0$
Thus $\exists R: \left|\frac{1}{p(z)}\right| < 1 \quad \forall z: |z| > R$.
Let $M = \max\limits_{|z|\le R} \frac{1}{|p(z)|}$. Then $\forall z \in \mathbb{C} \quad \frac{1}{|p(z)|} \le \max(1, M)$.
So, by Liouville, $\frac{1}{p(z)} \equiv \text{const}$ – contradiction! ∎

4)

Karl Weierstrass

Theorem (Weierstrass).
Let (f_n) be a sequence of analytic functions in a region Ω. Assume that f_n converges locally uniformly to a function f. Then
1) $f \in \mathcal{A}(\Omega)$
2) $f_n^{(k)} \to f^{(k)}$ locally uniformly in Ω.

Proof. 1) To prove that $f \in \mathcal{A}(\Omega)$, it is enough to

Show that $\forall z_0 \in \Omega \;\; \exists r > 0 : f \in A(\overline{B(z_0, r)})$.

Fix $z_0 \in \Omega$, $\quad r < \operatorname{dist}(z_0, \partial\Omega)$. $\quad \overline{B(z_0, r)} \subset \Omega \Rightarrow$

$f_n \rightrightarrows f$ uniformly on $B(z_0, r)$

Let $\gamma \subset B(z_0, r)$ – a closed arc. Then, by uniform convergence,

$\oint_\gamma f\,dz = \lim_{n\to\infty} \oint_\gamma f_n\,dz = 0$. By Morera, $f \in A(\Omega)$.

2) Fix $z_0 \in \Omega$. Take r: $\;2r < \operatorname{dist}(z_0, \partial\Omega)$.

Then $\overline{B(z_0, 2r)} \subset \Omega$,

$f_n \rightrightarrows f$ uniformly on $\overline{B(z_0, r)}$, let $a_n := \sup_{\overline{B(z_0, 2r)}} |f_n(z) - f(z)| \to 0$.

Then, by Cauchy inequalities, $\forall z \in B(z_0, r)$, $B(z, r) \subset B(z_0, 2r) \subset \Omega$

$|f_n^{(k)}(z) - f^{(k)}(z)| = |(f_n - f)^{(k)}(z)| \le \underset{\text{Cauchy for } B(z,r)}{k!\,a_n r^{-k}} \to 0$, so

$f_n^{(k)} \rightrightarrows f^{(k)}$ uniformly on $B(z_0, r)$. $\blacksquare$

Restatement: $f_n \in A(\Omega)$, $\sum_{n=1}^{\infty} f_n(z) = f(z)$ locally uniformly

Then $f \in A(\Omega)$ and $\forall k$: $\sum_{n=1}^{\infty} f_n^{(k)}(z) = f^{(k)}(z)$ locally uniformly.